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ON EQUILIBRIA OF EXCESS DEMAND CORRESPONDENCES

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ABSTRACT

A new lemma on the existence of maximal elements of binary relations is proved and applied to a revealed preference relation on price vectors. The resulting maximal elements are equilibrium prices. This technique allows one to generalize results of Aliprantis and Brown [1982], Neuefeind [1980], and Geistdoerfer-Florenzano [1982].

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Introduction.

In this note we present a new technique for generalizing some well known theorems on the existence of equilibrium prices for excess demand correspondences. The basic result in this area is due to Debreu [1956], Gale [1955], Kuhn [1956], and Nikaido [1956]. Recent variations are due to Grandmont [1977] and Neuefeind [1980] who introduce boundary conditions on the behavior of excess demand, and by Geistdoerfer-Florenzano [1982] who allows a weakening of Walras' law and the continuity condition. Aliprantis and Brown [1982] extend the equilibrium theorem to certain infinite-dimensional spaces for the case of excess demand functions (singleton-valued correspondences). This paper extends the results of Geistdoerfer-Florenzano to correspondences satisfying a boundary condition. The method of proof is based on the fact noted by Arrow and Hurwicz [1958] that if the excess demand correspondence is singleton-valued and satisfies a gross substitutability condition, then the equilibrium price vector minimizes a revealed preference relation. Following Aliprantis and Brown we make use of a generalization of a lemma of Sonnenschein [1971] to prove the existence of equilibrium prices by means of maximizing a revealed preference type of relation for price vectors.

* I have benefited greatly from the comments of Don Brown and Ed Green.

We then use a separating hyperplane argument to extend their result to correspondences. We motivate the technique with a discussion of the trivial case of the one person pure exchange economy.

The One Person Pure Exchange Economy.

Consider the case of an individual with a well behaved utility function and a fixed endowment ω of commodities. All prices are taken to lie in the unit simplex. At prices p the individual receives income $p \cdot \omega$ and demands the commodity vector $h(p)$. Define the excess demand function E by $E(p) = h(p) - \omega$. An equilibrium price vector $\bar{p}$ satisfies $E(\bar{p}) = 0$, i.e., $h(p) = \omega$. It is well-known from the theory of compensated demand curves that for any price vector p with relative prices different from $\bar{p}$ the commodity vector $h(p)$ will be preferred to $h(\bar{p}) = \omega$. By the nature of the budget constraint, ω is always affordable, so it follows from the weak axiom of revealed preference that $h(p)$ cannot be afforded at prices $\bar{p}$, that is, we must have that $\bar{p} \cdot h(p) > \bar{p} \cdot \omega$. In terms of excess demands this becomes $\bar{p} \cdot E(p) > 0$. If we define the binary relation U between price vectors by q U p if $q \cdot E(p) > 0$ then the equilibrium price vector is the largest element of U. Conversely, let q be a maximal element of U, i.e., for all other prices p, not p U q. Then $p \cdot E(q) \leq 0$ for all p. It then follows from Walras' law that $E(q) = 0$. Thus if the weak axiom of revealed preference holds the equilibrium price is unique. This approach reduces the search for equilibrium prices to the search for maximal elements of a revealed preference relation.

When there is more than one consumer, the excess demand function need not obey the weak axiom of revealed preference. Remarkably enough, the same approach works even in this case, though the equilibrium need not be unique. Thus we next present a lemma on the existence of maximal elements of binary relations.

Maximal Elements of Binary Relations.

A binary relation on a set K can be described in terms of a correspondence $U : K \to\to K$, where $U(p)$ is interpreted to be the set elements of K which are larger than p (the upper contour set). A *U-maximal* element of K is a $\bar{p} \in K$ for which $U(\bar{p}) = \emptyset$. With each binary relation U we associate the correspondence $U^{-1} : K \to\to K$ by $q \in U^{-1}(p)$ if and only if $p \in U(q)$. The following lemma is a strengthening of results of Fan [1961, Lemma 4] and Sonnenschein [1971, Theorem 4].

Lemma. Let K be a compact convex nonempty subset of a Hausdorff topological vector space and let $U : K \to\to K$ be a binary relation satisfying for all $p \in K$

(i) $x \notin \text{co}\, U(p)$ (where co denotes convex hull).

(ii) if $q \in U^{-1}(p)$, then there is a p' (possibly $= p$) such that $q \in \text{int}\, U^{-1}(p')$.

Then the set of U-maximal elements of K is compact and nonempty.

Proof. By definition, the set of U-maximal elements is just $\bigcap_{p \in K} (K \setminus U^{-1}(p))$. By hypothesis (ii),

$$\bigcap_{p \in K} (K \setminus U^{-1}(p)) = \bigcap_{p' \in K} (K \setminus \text{int}\, U^{-1}(p')).$$

This latter intersection is clearly compact, being the intersection of compact sets.

We now show that it is nonempty. For each p, put $F(p) = K \setminus (\text{int}\, U^{-1}(p))$. As noted above each $F(p)$ is compact. We first claim that if $q \in \text{co}\,\{p^i : i = 1,\dots,n\}$, then $q \in \bigcup_{i=1}^{n} F(p^i)$: Suppose that $q \notin \bigcup_{i=1}^{n} F(p^i)$. Then $q \in U^{-1}(p^i)$ for all i, so $p^i \in U(q)$ for all i. But then $q \in \text{co}\,\{p^i\} \subset \text{co}\, U(q)$, which violates (i). It then follows from the Knaster-Kuratowski-Mazurkiewicz lemma as extended by Fan [1961, Lemma 1] that $\bigcap_{p \in K} F(p) \neq \emptyset$.

Q.E.D.

Excess Demand Correspondences.

Let $\mathbb{R}^n$ be the commodity space. Denote by $\mathbb{R}^n_+$ the positive cone of $\mathbb{R}^n$, i.e., $\mathbb{R}^n_+ = \{z \in \mathbb{R}^n : z_i \geq 0 \text{ for } i = 1,\dots n\}$, and N denote the negative cone, $-\mathbb{RE}^n_+$. Let

$\Delta = \{p \in \mathbb{R}^n : \sum_{i=1}^{n} p_i = 1,\ p_i \geq 0 \quad i = 1,\dots,n\}$ and

$S = \{p \in \mathbb{R}^n : p \gg 0\}$ where $p \gg q$ means $p_i > q_i$, $i = 1,\dots,n$. The domain of the excess demand correspondence will be denoted by D, and it will be either equal either to S or to Δ.

A correspondence $E : D \to\to \mathbb{R}^n$ satisfies the *weak weak form of Walras' law* if for each p, (WWWL) $p \cdot z \leq 0$ for some $z \in E(p)$.

A correspondence $E : D \twoheadrightarrow \mathbb{R}^n$ satisfies the strong form of Walras' law if for each p,

(SWL) $p \cdot z = 0$ for all $z \in E(p)$.

In a private ownership economy, Walras' law will be satisfied if all consumers spend exactly all of their income.

A correspondence $E : S \twoheadrightarrow \mathbb{R}^n$ is said to satisfy the boundary condition (B) if the following holds.

(B) if for every $q \in \Delta$, there is a $p_q \in \Delta$ such that for every sequence $\{q_\alpha\} \subset S$ with $q_\alpha \to q \in \Delta \backslash S$, there is a subsequence $\{q_\beta\}$ such that for all β, $p_q \cdot z > 0$ for all $z \in E(q_\beta)$.

Boundary condition (B) is weaker than the condition introduced by Neuefeind [1980,Lemma 2], which requires p_q to be independent of q. It is however, stronger than the condition used by Aliprantis and Brown [1982] or Grandmont [1977]. Their condition allows p_q to depend on the sequence as well as its limit.

We say that the correspondence $E : D \twoheadrightarrow \mathbb{R}^n$ is upper demi-continuous if for each open half-space H in $\mathbb{R}^n$, $E^+[H] \equiv \{p : E(p) \subset H\}$ is open in D. (This terminology is from Browder [1967]. For closed-valued correspondences this definition is equivalent to what Geistdoerfer-Florenzano calls upper hemi-continuity. For finite-dimensional spaces every neighborhood of a point contains a neighborhood which is a finite intersection of open half-spaces, in which case demi-continuity reduces to hemi-continuity in the sense of Hildenbrand [1974].)

A free disposal equilibrium price of the correspondence $E : D \twoheadrightarrow \mathbb{R}^n$ is a price p such that $E(p) \cap N \neq \emptyset$. An equilibrium price p is a price $p \in S$ such that $0 \in E(p)$.

Existence of Equilibrium Prices.

We will treat two cases. In the first case the excess demand correspondence is assumed to be defined for all nonnegative prices and to satisfy the weak weak Walras' law and to take on only compact convex values. The existence of free disposal equilibrium prices is established. In the second case, the excess demand correspondence is defined only for strictly positive prices and is assumed to exhibit nice boundary behavior. In particular, we assume boundary condition (B). The strong form of Walras' law is assumed, though the values of the correspondence are only assumed to be closed and convex. Existence of equilibrium prices rather than just free disposal equilibrium prices is established. In both cases the excess demand correspondence is assumed be upper demi-continuous and to be homogeneous of degree zero, so that only normalized prices are considered.

The outline of the proof in each case is as follows. We define a binary relation between prices based on the revealed preference argument for the one person case and show that the maximal elements must be equilibrium prices. This part of the proof uses Walras' law and the fact that the excess demand correspondence assumes closed convex values, and is based on a separating hyperplane argument. We then show that the relation satisfies the hypotheses of the lemma, and so it has a nonempty compact set of maximal elements. This part of the proof uses Walras' law, demi-continuty and the boundary condition. One can view this approach as an attempt to reduce the general case to the one person pure exchange economy.

<u>Theorem 1.</u> Let $E : \Delta \rightarrow\rightarrow \mathbb{R}^n$ be upper demi-continuous with nonempty compact convex values and satisfy the weak weak form of Walras' law. Then the set of free disposal equilibrium prices for E is compact and nonempty.

<u>Proof.</u> For each $p \in \Delta$ set

$$U(p) = \{q : q \cdot z > 0 \text{ for all } z \in E(p)\}.$$

Then U satisfies the hypotheses of the lemma: For U(p) is clearly convex for each p and by Walras' law $p \notin U(p)$. Also $U^{-1}(p)$ is open for each p:

For if $q \in U^{-1}(p)$ we have that $p \cdot z > 0$ for all $z \in E(q)$. Then since E is upper demi-continuous $E^{+}[\{x : p \cdot x > 0\}]$ is a neighborhood of q in $U^{-1}(p)$.

Now p is U-maximal if and only if

$$\text{for each } q \in \Delta, \text{ there is a } z \in E(p) \text{ with } q \cdot z \leq 0. \qquad (*)$$

We now show that $E(p) \cap N \neq \emptyset$ if and only if (*). Suppose $z \in E(p) \cap N$. Then $q \cdot z \leq 0$ for all $q \in \Delta$, so (*) holds. If on the other hand, $E(p) \cap N = \emptyset$, then we can strictly separate E(p) which is compact and convex from N which is closed and convex (Schaefer [1971, 9.2]). That is there exists some $q \in \mathbb{R}^n$ and some $c \in \mathbb{R}$ satisfying

$$x \in N \text{ and } z \in E(p) \text{ imply } q \cdot z > c > q \cdot x$$

Since N is the negative cone, $c > 0$ and $q \in \mathbb{R}^n_+$. Thus $q \cdot z > 0$ for all $z \in E(p)$, and (*) does not hold.

It thus follows from the lemma that the set of free disposal equilibrium prices is compact and nonempty.

Q.E.D.

<u>Theorem 2.</u> Let $E : S \rightarrow\rightarrow \mathbb{R}^n$ be upper demi-continuous with nonempty, closed, convex values and satisfy the strong from of Walras' law and the boundary condition (B) (relative to S and Δ):

(SWL) $\quad p \cdot z = 0 \qquad$ for all $z \in E(p)$.

(B) if for every $q \in \Delta$, there is a $p_q \in \Delta$ such that for every net $\{q_\alpha\} \subset S$ with $q_\alpha \rightarrow q \in \Delta \backslash S$, there is a subnet $\{q_\beta\}$ such that for all β, $p_q \cdot z > 0$ for all $z \in E(q_\beta)$.

Then the set of equilibrium prices for E is compact and nonempty.

<u>Proof.</u> Define the binary relation U on Δ by

$$p \in U(q) \quad \text{if} \quad \begin{array}{l} p \cdot z > 0 \quad \text{for all } z \in E(q) \quad \text{and} \quad p,q \in S \\ \qquad\qquad \text{or} \\ p \in S,\ q \in \Delta\backslash S. \end{array}$$

We first show that the U-maximal elements are precisely the equilibrium prices. First suppose that $\bar{p}$ is U-maximal, i.e., $U(\bar{p}) = \emptyset$. Since $U(p) = S$ for all $p \in \Delta\backslash S$, we have that $\bar{p} \in S$. Since $\bar{p} \in S$ and $U(\bar{p}) = \emptyset$ we have

$$\text{for each } q \in S, \text{ there is a } z \in E(\bar{p}) \text{ with } q \cdot z \leq 0. \qquad (1)$$

We now show that (1) implies $0 \in E(\bar{p})$. Suppose by way of contradiction that $0 \notin E(\bar{p})$. Then we can strictly separate $\{0\}$ which is compact and convex from $E(\bar{p})$ which is closed and convex with $\tilde{p} \in \mathbb{R}^n$. Thus $\tilde{p} \cdot z > 0$ for all $z \in E(\bar{p})$. Put $p_\lambda = \lambda\tilde{p} + (1 - \lambda)\bar{p}$. Then for $z \in E(\bar{p})$, $p_\lambda \cdot z = \lambda\tilde{p} \cdot z + (1 - \lambda)\bar{p} \cdot z = \lambda\tilde{p} \cdot z > 0$ for $\lambda > 0$. (Recall that $\bar{p} \cdot z = 0$ for $z \in E(\bar{p})$ by Walras' law.) For $\lambda > 0$ small enough, $p_\lambda \gg 0$ so that the normalized price vector $\bar{p}_\lambda = (\sum (p_\lambda)_i)^{-1} p_\lambda \in S$ and $\bar{p}_\lambda \cdot z > 0$ for all $z \in E(\bar{p})$, which violates (1).

Conversely, if $\bar{p}$ is an equilibrium price, then $0 \in E(\bar{p})$ and since $p \cdot 0 = 0$ for all p, $U(\bar{p}) = \emptyset$.

We now verify that U satisfies the hypotheses of the lemma:

(ia) $p \notin U(p)$: For $p \in S$ this follows from Walras' law. For $p \in \Delta\backslash S$, $p \notin S = U(p)$.

(ib) $U(p)$ is convex: For $p \in S$, let $q^1, q^2 \in U(p)$, i.e. $q^1 \cdot z > 0$, $q^2 \cdot z > 0$ for $z \in E(p)$. Then $[\lambda q^1 + (1 - \lambda)q^2] \cdot z > 0$

as well. For $p \in \Delta\backslash S$, $U(p) = S$ which is convex.

(ii) If $q \in U^{-1}(p)$, then there is a p' with $q \in \text{int } U^{-1}(p')$: There are two cases: (a) $q \in S$ and (b) $q \in \Delta\backslash S$.

(a) $q \in S \cap U^{-1}(p)$. Then $p \cdot z > 0$ for all $z \in E(q)$. Let H be the open half space $\{x : p \cdot x > 0\}$. Then by upper demi-continuity, $E^+[H]$ is a neighborhood of q contained in $U^{-1}(p)$.

(b) $q \in (\Delta\backslash S) \cap U^{-1}(p)$. By boundary condition (B) $q \in \text{int } U^{-1}(p_q)$: Suppose $q \notin \text{int } U^{-1}(p_q)$. Then there is a net $q_\alpha \to q$ with $q_\alpha \notin U^{-1}(p_q)$ for all α. This implies that $q_\alpha \in S$ and so there is some $z_\alpha \in E(q_\alpha)$ with $p_q \cdot z_\alpha \leq 0$. This violates (B).

Q.E.D.

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